GLOBAL EDITION

AN INTRODUCTION TO ANALYSIS

FOURTH EDITION

WILLIAM R. WADE

An Introduction to Analysis

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Fourth Edition
Global Edition

William R. Wade
University of Tennessee

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To Cherri, Peter, and Benjamin

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Contents

Preface

This text provides a bridge from "sophomore" calculus to graduate courses which use analytic ideas (e.g., real and complex analysis, partial and ordinary differential equations, numerical analysis, fluid mechanics, and differential geometry). For a two-semester course, the first semester should end with Chapter 8. For a three-quarter course, the second quarter should begin in Chapter 6 and end somewhere in the middle of Chapter 11.

Our presentation is divided into two parts. The first half, Chapters 1 through 7 together with Appendices A and B, gradually introduces the central ideas of analysis in a one-dimensional setting. The second half, Chapters 8 through 14 together with Appendices C through F, covers multidimensional theory.

More specifically, Chapter 1 introduces the real number system as a complete, ordered field; Chapters 2 through 5 cover calculus on the real line; and Chapters 6 and 7 discuss infinite series, including uniform and absolute convergence. Chapter 8 gives a short introduction to the algebraic structure of $\mathbf{R}^n$, including the connection between linear functions and matrices.

At that point instructors have two options. They can cover Chapter 9 to explore topology and convergence in the concrete Euclidean space setting, or they can cover these same concepts in the abstract metric space setting (Chapter 10). Since either of these options provides the necessary foundation for the rest of the book, instructors are free to choose the approach they feel best suits their aims.

With this background material out of the way, Chapters 11 through 13 develop the machinery and theory of vector calculus. Chapter 14 gives a short introduction to Fourier series, including summability and convergence of Fourier series, growth of Fourier coefficients, and uniqueness of trigonometric series.

Separating the one-dimensional from the n-dimensional material is not the most efficient way to present the material, but it does have two advantages. The more abstract, geometric concepts can be postponed until students have been given a thorough introduction to analysis on the real line. And, students have two chances to master some of the deeper ideas of analysis (e.g., convergence of sequences, limits of functions, and uniform continuity).

We have made this text flexible in another way by including core material and enrichment material. The core material provides a foundation for the typical one-year course in analysis. Besides making the book a better reference, the enrichment material has been included for two other reasons: curious students can use it to delve deeper into the core material or as a jumping off place to pursue more general topics, and instructors can use it to supplement their course or to add variety from year to year.

Enrichment and optional materials are marked with an asterisk. Exercises which use enrichment material are also marked with an asterisk, and the material needed to solve them is cited in the Answers and Hints section. To make course planning easier, each enrichment section begins with a statement which indicates whether that section uses material from any other enrichment section. Since no core material depends on enrichment material, any of the latter can be skipped without loss in the integrity of the course.

Most enrichment sections (5.5, 5.6, 6.5, 6.6, 7.5, 9.3, 11.6, 12.6, 14.1) are independent and can be covered in any order after the core material which precedes them has been dealt with. Sections 9.8 and 12.5 require 9.3, Section 14.3 requires 5.5 only to establish Lemma 14.24. This result can easily be proved for continuously differentiable functions, thereby avoiding mention of functions of bounded variation. In particular, the key ideas in Section 14.3 can be covered without the background material from Section 5.5 anytime after finishing Chapter 7.

Since for many students this is the last (for some the only) place to see a rigorous development of vector calculus, we focus our attention on classical, nitty-gritty analysis. By avoiding abstract concepts such as vector spaces and the Lebesgue integral, we have room for a thorough, comprehensive introduction. We include sections on improper integration, the gamma function, Lagrange multipliers, the Inverse and Implicit Function Theorem, Green's Theorem, Gauss's Theorem, Stokes's Theorem, and a full account of the change of variables formula for multiple Riemann integrals.

We assume the reader has completed a three-semester or four-quarter sequence in elementary calculus. Because many of our students now take their elementary calculus in high school (where theory may be almost nonexistent), we assume that the reader is familiar only with the mechanics of calculus, i.e., can differentiate, integrate, and graph simple functions of the form $y = f(x)$ or $z = f(x, y)$. We also assume the reader has had an introductory course in linear algebra, i.e., can add, multiply, and take determinants of matrices with real entries, and are familiar with Cramer's Rule. (Appendix C, which contains an exposition of all definitions and theorems from linear algebra used in the text, can be used as review if the instructor deems it necessary.)

Always we emphasize the fact that the concepts and results of analysis are based on simple geometric considerations and on analogies with material already known to the student. The aim is to keep the course from looking like a collection of tricks and to share enough of the motivation behind the mathematics so that students are prepared to construct their own proofs when asked. We begin complicated proofs with a short paragraph (marked STRATEGY:) which shows why the proof works; for example, the Archimedean Principle (Theorem 1.16), Density of Rationals (Theorem 1.18), Cauchy's Theorem (Theorem 2.29), Change of Variables in **R** (Theorem 5.34), Riemann's Theorem about rearrangements (Theorem 6.29), the Implicit Function Theorem (Theorem 11.47), the Borel Covering Lemma (Lemma 9.26), and the fact that a curve is smooth when $\phi' \neq \mathbf{0}$ (Remark 13.10). We precede abstruse definitions or theorems with a short paragraph which describes, in simple terms, what behavior we are examining, and why (e.g., Cauchy sequences, one-sided limits, upper and lower Riemann sums, the Integral Test, Abel's Formula, uniform convergence, the total derivative, compact sets, differentiable curves and surfaces, smooth curves, and orientation equivalence). And we include examples to show why each hypothesis of a major theorem is necessary (e.g., the Nested Interval Property, the Bolzano–Weierstrass Theorem, the Mean Value Theorem, the Heine–Borel Theorem, the Inverse Function Theorem, the existence of exact differentials, and Fubini's Theorem).

Each section contains a collection of exercises which range from very elementary (to be sure the student understands the concepts introduced in that section) to more challenging (to give the student practice in using these concepts to expand the theory). To minimize frustration, some of the more difficult exercises have several parts which serve as an outline to a solution of the problem. To keep from producing students who know theory but cannot apply it, each set of exercises contains a mix of computational and theoretical assignments. (Exercises which play a prominent role later in the text are marked with a box. These exercises are an integral part of the course and all of them should be assigned.)

Since many students have difficulty reading and understanding mathematics, we have paid close attention to style and organization. We have consciously limited the vocabulary, kept notation consistent from chapter to chapter, and presented the proofs in a unified style. Individual sections are determined by subject matter, not by length of lecture, so that students can comprehend related results in a larger context. Examples and important remarks are numbered and labeled so that students can read the text in small chunks. (Many of these, included for the student's benefit, need not be covered in class.) Paragraphs are short and focused so that students are not overwhelmed by long-winded explanations. To help students discern between central results and peripheral ones, the word *Theorem* is used relatively sparingly; preliminary results and results which are only used in one section are called Remarks, Lemmas, and Examples. And we have broken with tradition by stating definitions explicitly with an "if and only if." (How can we chide our students for using the converse of a result when it appears that we do so about half the time we apply a definition?)

NEW TO THIS EDITION

We have changed many of the computational exercises so that the answers are simpler and easier to obtain. We have replaced most of the beginning calculus–style exercises with slightly more conceptual exercises that emphasize the same ideas, but at a higher level. We have added many theoretical exercises of medium difficulty. We have scattered true-false questions throughout the first six chapters. These are designed to confront common misconceptions that some students tend to acquire at this level. We have gathered introductory material that was scattered over several sections into a new section entitled *Introduction*. This section includes two accessible examples about why proof is necessary and why we cannot always trust what we see. We have reduced the number of axioms from four to three by introducing the Completeness Axiom first, and using it to prove the Well-Ordering Principle and the Principle of Mathematical Induction. We have moved Taylor's Formula from Chapter 7 to Chapter 4 to offer another example of the utility of the Mean Value Theorem. We have given the Heine–Borel Theorem its own section and included several exercises designed to give students practice in making a local condition on a compact set into a global one. We have reorganized Section 12.1 (Jordan regions) to simplify the presentation and make it easier to teach. We have omited Chapter 15, And we have corrected a number of misprints.

We wish to thank Mr. P. W. Wade and Professors S. Fridli, G. S. Jordan, Mefharet Kocatepe, J. Long, M. E. Mays, M. S. Osborne, P. W. Schaefer, F. E. Schroeck, and Ali Sinan Sertoz, who carefully read parts of the first edition and made many valuable suggestions and corrections. Also, I wish to express my gratitude to Ms. C. K. Wade for several lively discussions of a pedagogical nature, which helped shape the organization and presentation of this material. I wish to thank Der-Chen Chang (Georgetown University), Wen D. Chang (Alabama State), Patrick N. Dowling (Miami University), Jeffery Ehme (Spelman College), Dana S. Fine (University of Massachusetts-Dartmouth), Stephen Fisher (Northwestern University), Scott Fulton (Clarkson University), Kevin Knudson (Mississippi State University), Maria Nogin (California State University- Fresno), Gary Weiss (University of Cincinnati), Peter Wolfe (University of Maryland), and Mohammed Yahdi (Ursinus College) who looked at the fourth edition while it was in manuscript form and did some pre-revision reviews. I wish to thank Professor Stan Perrine (Charleston Southern University) for checking the penultimate version of the manuscript for accuracy and typographical errors. Finally, I wish to make special mention of Professor Lewis Lum, (Portland State University) who continues to make many careful and perspicuous comments about style, elegance of presentation, and level of rigor which have found their way into this fourth edition.

If there remain any typographical errors, I plan to keep an up-to-date list at my Web site. If you find errors which are not listed at that site, please contact me via e-mail at iwade@utk.edu.

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CHAPTER 1

The Real Number System

You have already had several calculus courses in which you evaluated limits, differentiated functions, and computed integrals. You may even remember some of the major results of calculus, such as the Chain Rule, the Mean Value Theorem, and the Fundamental Theorem of Calculus. Although you are probably less familiar with multivariable calculus, you have taken partial derivatives, computed gradients, and evaluated certain line and surface integrals.

In view of all this, you must be asking: Why another course in calculus? The answer to this question is twofold. Although some proofs may have been presented in earlier courses, it is unlikely that the subtler points (e.g., completeness of the real numbers, uniform continuity, and uniform convergence) were covered. Moreover, the skills you acquired were mostly computational; you were rarely asked to prove anything yourself. This course develops the theory of calculus carefully and rigorously from basic principles and gives you a chance to learn how to construct your own proofs. It also serves as an introduction to analysis, an important branch of mathematics which provides a foundation for numerical analysis, functional analysis, harmonic analysis, differential equations, differential geometry, real analysis, complex analysis, and many other areas of specialization within mathematics.

1.1 INTRODUCTION

Every rigorous study of mathematics begins with certain undefined concepts, primitive notions on which the theory is based, and certain postulates, properties which are assumed to be true and given no proof. Our study will be based on the primitive notions of real numbers and sets, which will be discussed in this section.

We shall use standard notation for sets and real numbers. For example, **R** or $(-\infty, \infty)$ represents the set of *real numbers*, $\emptyset$ represents the *empty set* (the set with no elements), $a \in A$ means that a is an *element of* A, and $a \notin A$ means that a is *not* an element of A. We can represent a given finite set in two ways. We can list its elements directly, or we can describe it using sentences or equations. For example, the set of solutions to the equation $x^2 = 1$ can be written as

$$\{1, -1\} \quad \text{or} \quad \{x : x^2 = 1\}.$$

A set A is said to be a *subset* of a set B (notation: $A \subseteq B$) if and only if every element of A is also an element of B. If A is a subset of B but there is at least one element $b \in B$ that does not belong to A, we shall call A a *proper subset* of B (notation: $A \subset B$). Two sets A and B are said to be *equal* (notation: $A = B$)

if and only if $A \subseteq B$ and $B \subseteq A$. If A and B are not equal, we write $A \neq B$. A set A is said to be *nonempty* if and only if $A \neq \emptyset$.

The *union* of two sets A and B (notation: $A \cup B$) is the set of elements x such that x belongs to A or B or both. The *intersection* of two sets A and B (notation: $A \cap B$) is the set of elements x such that x belongs to both A and B. The *complement* of B *relative to* A (notation: $A \setminus B$, sometimes B^c if A is understood) is the set of elements x such that x belongs to A but does not belong to B. For example,

$$\{-1, 0, 1\} \cup \{1, 2\} = \{-1, 0, 1, 2\}, \qquad \{-1, 0, 1\} \cap \{1, 2\} = \{1\},$$
$$\{1, 2\} \setminus \{-1, 0, 1\} = \{2\} \quad \text{and} \quad \{-1, 0, 1\} \setminus \{1, 2\} = \{-1, 0\}.$$

Let X and Y be sets. The *Cartesian product* of X and Y is the set of *ordered pairs* defined by

$$X \times Y := \{(x, y) : x \in X, y \in Y\}.$$

(The symbol $:=$ means "equal by definition" or "is defined to be.") Two points (x, y), $(z, w) \in X \times Y$ are said to be *equal* if and only if $x = z$ and $y = w$.

Let X and Y be sets. A *relation* on $X \times Y$ is any subset of $X \times Y$. Let $\mathcal{R}$ be a relation on $X \times Y$. The *domain* of $\mathcal{R}$ is the collection of $x \in X$ such that (x, y) belongs to $\mathcal{R}$ for some $y \in Y$. The *range* of $\mathcal{R}$ is the collection of $y \in Y$ such that (x, y) belongs to $\mathcal{R}$ for some $x \in X$. When $(x, y) \in \mathcal{R}$, we shall frequently write $x\mathcal{R}y$.

A *function* f from X into Y (notation: $f : X \to Y$) is a relation on $X \times Y$ whose domain is X (notation: $\mathrm{Dom}(f) := X$) such that for each $x \in X$ there is a *unique* (one and only one) $y \in Y$ that satisfies $(x, y) \in f$. If $(x, y) \in f$, we shall call y the *value* of f at x (notation: $y = f(x)$ or $f : x \longmapsto y$) and call x a *preimage* of y under f. We said *a* preimage because, in general, a point in the range of f might have more than one preimage. For example, since $\sin(k\pi) = 0$ for $k = 0, \pm 1, \pm 2, \ldots$, the value 0 has infinitely many preimages under $f(x) = \sin x$.

If f is a function from X into Y, we will say that f is *defined* on X and call Y the *codomain* of f. The *range* of f is the collection of all values of f; that is, the set $\mathrm{Ran}(f) := \{y \in Y : f(x) = y \text{ for some } x \in X\}$. In general, then, the range of a function is a subset of its codomain and each $y \in \mathrm{Ran}(f)$ has one or more preimages. If $\mathrm{Ran}(f) = Y$ and each $y \in Y$ has exactly one preimage, $x \in X$, under f, then we shall say that $f : X \to Y$ *has an inverse*, and shall define the *inverse function* $f^{-1} : Y \to X$ by $f^{-1}(y) := x$, where $x \in X$ satisfies $f(x) = y$.

At this point it is important to notice a consequence of defining a function to be a set of ordered pairs. By the definition of equality of ordered pairs, two functions f and g are equal if and only if they have the same domain, and same values; that is, $f, g : X \to Y$, and $f(x) = g(x)$ for all $x \in X$. If they have different domains, they are different functions.

For example, let $f(x) = g(x) = x^2$. Then $f : [0, 1) \to [0, 1)$ and $g : (-1, 1) \to [0, 1)$ are two different functions. They both have the same range, $[0, 1)$, but each $y \in (0, 1)$ has exactly one preimage under f, namely $\sqrt{y}$, and two preimages under g, namely $\pm\sqrt{y}$. In particular, f has an inverse function, $f^{-1}(x) = \sqrt{x}$, but g does not. Making distinctions like this will actually make our life easier later in the course.

For the first half of this course, most of the concrete functions we consider will be *real-valued functions of a real variable* (i.e., functions whose domains and ranges are subsets of **R**). We shall often call such functions simply *real functions*.

You are already familiar with many real functions.

1) The *exponential function* $e^x : \mathbf{R} \to (0, \infty)$ and its inverse function, the *natural logarithm*

$$\log x := \int_1^x \frac{dt}{t},$$

defined and real-valued for each $x \in (0, \infty)$. (Although this last function is denoted by $\ln x$ in elementary calculus texts, most analysts denote it, as we did just now, by $\log x$. We will follow this practice throughout this text. For a more constructive definition, see Exercise 4.5.5.)

2) The *trigonometric functions* (whose formulas are) represented by $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$, and the inverse trigonometric functions $\arcsin x$, $\arccos x$, and $\arctan x$ whose ranges are, respectively, $[-\pi/2, \pi/2]$, $[0, \pi]$, and $(-\pi/2, \pi/2)$.

3) The *power functions* x^α, which can be defined constructively (see Appendix A.10 and Exercise 3.3.11) or by using the exponential function:

$$x^\alpha := e^{\alpha \log x}, \qquad x > 0, \quad \alpha \in \mathbf{R}.$$

We assume that you are familiar with the various algebraic laws and identities that these functions satisfy. A list of the most widely used trigonometric identities can be found in Appendix B. The most widely used properties of the power functions are $x^0 = 1$ for all $x \neq 0$; $x^n = x \cdot \ldots \cdot x$ (there are n factors here) when $n = 1, 2, \ldots$ and $x \in \mathbf{R}$; $x^\alpha > 0$, $x^\alpha \cdot x^\beta = x^{\alpha+\beta}$, and $(x^\alpha)^\beta = x^{\alpha \cdot \beta}$ for all $x > 0$ and $\alpha, \beta \in \mathbf{R}$; $x^\alpha = \sqrt[m]{x}$ when $\alpha = 1/m$ for some $m \in= 1, 2, \ldots$ and the indicated root exists and is real; and $0^\alpha := 0$ for all $\alpha > 0$. (The symbol 0^0 is left undefined because it is indeterminate [see Example 4.31].)

We also assume that you can differentiate algebraic combinations of these functions using the basic formulas $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, and $(e^x)' = e^x$, for $x \in \mathbf{R}$; $(\log x)' = 1/x$ and $(x^\alpha)' = \alpha x^{\alpha-1}$, for $x > 0$ and $\alpha \in \mathbf{R}$; and

$$(\tan x)' = \sec^2 x \quad \text{for } x \neq \frac{(2n+1)\pi}{2}, \qquad n \in \mathbf{Z}.$$

(You will have an opportunity to develop some of these rules in the exercises, e.g., see Exercises 4.2.9, 4.4.6, 4.5.3, 5.3.7, and 5.3.8.) Even with these assumptions, we shall repeat some material from elementary calculus.

We mentioned postulates in the opening paragraph. In the next two sections, we will introduce three postulates (containing a total of 13 different properties) which characterize the set of real numbers. Although you are probably already familiar with all but the last of these properties, we will use them to prove other equally familiar properties (e.g., in Example 1.4 we will prove that if $a \neq 0$, then $a^2 > 0$).

Why would we assume some properties and prove others? At one point, mathematicians thought that all laws about real numbers were of equal weight. Gradually, during the late 1800s, we discovered that many of the well-known laws satisfied by **R** are in fact consequences of others. The net result of this research is that the 13 properties listed below are considered to be fundamental properties describing **R**. All other laws satisfied by real numbers are secondary in the sense that they can be proved using these fundamental properties.

Why would we prove a law that is well known, perhaps even "obvious"? Why not just assume all known properties about **R** and proceed from there? We want this book to be reasonably self-contained, because this will make it easier for you to begin to construct your own proofs. We want the first proofs you see to be easily understood, because they deal with familiar properties that are unobscured by new concepts. But most importantly, we want to form a habit of proving all statements, even seemingly "obvious" statements.

The reason for this hard-headed approach is that some "obvious" statements are false. For example, divide an 8×8-inch square into triangles and trapezoids as shown on the left side of Figure 1.1. Since the 3-inch sides of the triangles perfectly match the 3-inch sides of the trapezoids, it is "obvious" that these triangles and trapezoids can be reassembled into a rectangle (see the right side of Figure 1.1). Or is it? The area of the square is $8 \times 8 = 64$ square inches but the area of the rectangle is $5 \times 13 = 65$ square inches. Since you cannot increase area by reassembling pieces, what looked right was in fact wrong. By computing slopes, you can verify that the rising diagonal on the right side of Figure 1.1 is, in fact, four distinct line segments that form a long narrow diamond which conceals that extra one square inch.

NOTE: *Reading a mathematics book is different from reading any other kind of book. When you see phrases like "you can verify" or "it is easy to see," you should use pencil and paper to do the calculations to be sure what we've said is correct.*

Here is another example. Grab a calculator and graph the functions $y = \log x$ and $y = \sqrt[100]{x}$. It is easy to see, using calculus, that $\log x$ and $\sqrt[100]{x}$ are both increasing and concave downward on $[0, \infty)$. Looking at the graphs (see Figure 1.2), it's "obvious" that $\log x$ is much larger than $\sqrt[100]{x}$ no matter how big x is. Or is it? Let's evaluate each function at e^{1000}: $\log(e^{1000}) = 1000 \log e = 1000$ is much smaller than $\sqrt[100]{e^{1000}} = e^{10} \approx 22,000$. Evidently, the graph of $y = \sqrt[100]{x}$

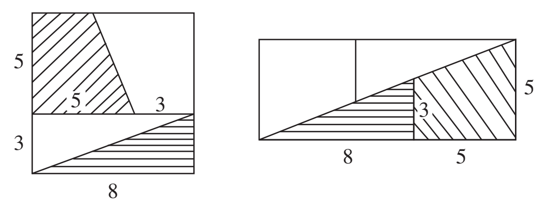

FIGURE 1.1

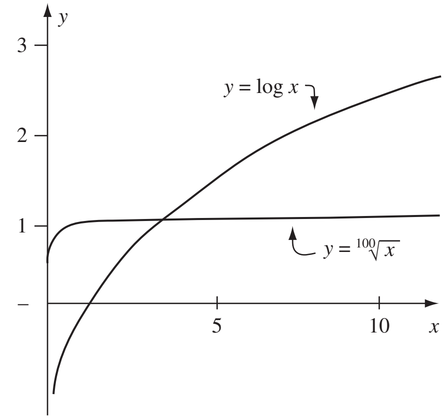

FIGURE 1.2

eventually crosses that of $y = \log x$. With a little calculus, you can prove that $\log x < \sqrt[100]{x}$ forever after that (see Exercise 4.4.6a).

What can be learned from these examples? We cannot always trust what we think we see. We must, as above, find some mathematical way of testing our perception, either verifying that it is correct, or rejecting it as wrong. This type of phenomenon is not a rare occurrence. You will soon encounter several other plausible statements that are, in fact, false. In particular, you must harbor a skepticism that demands proofs of all statements not assumed in postulates, even the "obvious" ones.

What, then, are you allowed to use when solving the exercises? You may use any property of real numbers (e.g., $2+3=5$, $2<7$, or $\sqrt{2}$ is irrational) without reference or proof. You may use any algebraic property of real numbers involving equal signs [e.g., $(x+y)^2 = x^2+2xy+y^2$ or $(x+y)(x-y) = x^2-y^2$] and the techniques of calculus to find local maxima or minima of a given function without reference or proof. After completing the exercises in Section 1.2, you may also use any algebraic property of real numbers involving inequalities (e.g., $0<a<b$ implies $0<a^x<b^x$ for all $x>0$) without reference or proof.

1.2 ORDERED FIELD AXIOMS

In this section we explore the algebraic structure of the real number system. We shall assume that the set of real numbers, **R**, is a field (i.e., that **R** satisfies the following postulate).

Postulate 1. [FIELD AXIOMS]. There are functions $+$ and $\cdot$, defined on $\mathbf{R}^2 := \mathbf{R} \times \mathbf{R}$, which satisfy the following properties for every $a, b, c \in \mathbf{R}$:

Closure Properties. $a+b$ and $a \cdot b$ belong to **R**.

Associative Properties. $a+(b+c) = (a+b)+c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Commutative Properties. $a+b = b+a$ and $a \cdot b = b \cdot a$.

Distributive Law. $a \cdot (b+c) = a \cdot b + a \cdot c$.

Existence of the Additive Identity. There is a unique element $0 \in \mathbf{R}$ such that $0 + a = a$ for all $a \in \mathbf{R}$.

Existence of the Multiplicative Identity. There is a unique element $1 \in \mathbf{R}$ such that $1 \neq 0$ and $1 \cdot a = a$ for all $a \in \mathbf{R}$.

Existence of Additive Inverses. For every $x \in \mathbf{R}$ there is a unique element $-x \in \mathbf{R}$ such that

$$x + (-x) = 0.$$

Existence of Multiplicative Inverses. For every $x \in \mathbf{R} \setminus \{0\}$ there is a unique element $x^{-1} \in \mathbf{R}$ such that

$$x \cdot (x^{-1}) = 1.$$

We note in passing that the word *unique* can be dropped from the statements in Postulate 1 (see Appendix A).

We shall usually denote $a + (-b)$ by $a - b$, $a \cdot b$ by ab, a^{-1} by $\frac{1}{a}$ or $1/a$, and $a \cdot b^{-1}$ by $\frac{a}{b}$ or a/b. Notice that by the existence of additive and multiplicative inverses, the equation $x + a = 0$ can be solved for each $a \in \mathbf{R}$, and the equation $ax = 1$ can be solved for each $a \in \mathbf{R}$ provided that $a \neq 0$.

From these few properties (i.e., from Postulate 1), we can derive all the usual algebraic laws of real numbers, including the following:

$$(-1)^2 = 1, \tag{1}$$

$$0 \cdot a = 0, \quad -a = (-1) \cdot a, \quad -(-a) = a, \qquad a \in \mathbf{R}, \tag{2}$$

$$-(a - b) = b - a, \qquad a, b \in \mathbf{R}, \tag{3}$$

and

$$a, b \in \mathbf{R} \quad \text{and} \quad ab = 0 \quad \text{imply} \quad a = 0 \quad \text{or} \quad b = 0. \tag{4}$$

We want to keep our attention sharply focused on analysis. Since the proofs of algebraic laws like these lie more in algebra than analysis (see Appendix A), we will not present them here. In fact, with the exception of the absolute value and the Binomial Formula, we will assume all material usually presented in a high school algebra course (including the quadratic formula and graphs of the conic sections).

Postulate 1 is sufficient to derive all algebraic laws of $\mathbf{R}$, but it does not completely describe the real number system. The set of real numbers also has an order relation (i.e., a concept of "less than").

Postulate 2. [Order Axioms]. There is a relation $<$ on $\mathbf{R} \times \mathbf{R}$ that has the following properties:

Trichotomy Property. Given $a, b \in \mathbf{R}$, one and only one of the following statements holds:

$$a < b, \quad b < a, \quad \text{or} \quad a = b.$$

Transitive Property. For $a, b, c \in \mathbf{R}$,

$$a < b \quad \text{and} \quad b < c \quad \text{imply} \quad a < c.$$

The Additive Property. For $a, b, c \in \mathbf{R}$,

$$a < b \quad \text{and} \quad c \in \mathbf{R} \quad \text{imply} \quad a + c < b + c.$$

The Multiplicative Properties. For $a, b, c \in \mathbf{R}$,

$$a < b \quad \text{and} \quad c > 0 \quad \text{imply} \quad ac < bc$$

and

$$a < b \quad \text{and} \quad c < 0 \quad \text{imply} \quad bc < ac.$$

By $b > a$ we shall mean $a < b$. By $a \leq b$ and $b \geq a$ we shall mean $a < b$ or $a = b$. By $a < b < c$ we shall mean $a < b$ and $b < c$. In particular, $2 < x < 1$ makes no sense at all.

WARNING. *There are two Multiplicative Properties, so every time you multiply an inequality by an expression, you must carefully note the sign of that expression and adjust the inequality accordingly*. For example, $x < 1$ does NOT imply that $x^2 < x$ unless $x > 0$. If $x < 0$, then by the Second Multiplicative Property, $x < 1$ implies $x^2 > x$.

We shall call a number $a \in \mathbf{R}$ *nonnegative* if $a \geq 0$ and *positive* if $a > 0$. Postulate 2 has a slightly simpler formulation using the set of positive elements as a primitive concept (see Exercise 1.2.11). We have introduced Postulate 2 as above because these are the properties we use most often.

The real number system $\mathbf{R}$ contains certain special subsets: the set of *natural numbers*

$$\mathbf{N} := \{1, 2, \ldots\},$$

obtained by beginning with 1 and successively adding 1s to form $2 := 1+1$, $3 := 2 + 1$, and so on; the set of *integers*

$$\mathbf{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

(*Zahl* is German for number); the set of *rationals* (or fractions or quotients)

$$\mathbf{Q} := \left\{\frac{m}{n} : m, n \in \mathbf{Z} \text{ and } n \neq 0\right\};$$

and the set of *irrationals*

$$\mathbf{Q}^c = \mathbf{R} \setminus \mathbf{Q}.$$

Equality in $\mathbf{Q}$ is defined by

$$\frac{m}{n} = \frac{p}{q} \quad \text{if and only if} \quad mq = np.$$

Recall that each of the sets **N, Z, Q**, and **R** is a proper subset of the next; that is,

$$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}.$$

For example, every rational is a real number (because $m/n := mn^{-1}$ is a real number by Postulate 1), but $\sqrt{2}$ is an irrational.

Since we did not really define **N** and **Z**, we must make certain assumptions about them. If you are interested in the definitions and proofs, see Appendix A.

1.1 Remark. We will assume that the sets **N** and **Z** satisfy the following properties.

i) If $n, m \in \mathbf{Z}$, then $n + m$, $n - m$, and mn belong to **Z**.
ii) If $n \in \mathbf{Z}$, then $n \in \mathbf{N}$ if and only if $n \geq 1$.
iii) There is no $n \in \mathbf{Z}$ that satisfies $0 < n < 1$.

Using these properties, we can prove that **Q** satisfies Postulate 1 (see Exercise 1.2.9).

We notice in passing that none of the other special subsets of **R** satisfies Postulate 1. **N** satisfies all but three of the properties in Postulate 1: **N** has no additive identity (since $0 \notin \mathbf{N}$), **N** has no additive inverses (e.g., $-1 \notin \mathbf{N}$), and only one of the nonzero elements of **N** (namely, 1) has a multiplicative inverse. **Z** satisfies all but one of the properties in Postulate 1: Only two nonzero elements of **Z** have multiplicative inverses (namely, 1 and -1). $\mathbf{Q}^c$ satisfies all but four of the properties in Postulate 1: $\mathbf{Q}^c$ does not have an additive identity (since $0 \notin \mathbf{R} \setminus \mathbf{Q}$), does not have a multiplicative identity (since $1 \notin \mathbf{R} \setminus \mathbf{Q}$), and does not satisfy either closure property. Indeed, since $\sqrt{2}$ is irrational, the sum of irrationals may be rational ($\sqrt{2} + (-\sqrt{2}) = 0$) and the product of irrationals may be rational ($\sqrt{2} \cdot \sqrt{2} = 2$).

Notice that any subset of **R** satisfies Postulate 2. Thus **Q** satisfies both Postulates 1 and 2. The remaining postulate, introduced in Section 1.3, identifies a property that **Q** does not possess. In particular, Postulates 1 through 3 distinguish **R** from each of its special subsets **N, Z, Q**, and $\mathbf{Q}^c$. These postulates actually characterize **R**; that is, **R** is the only set that satisfies Postulates 1 through 3. (Such a set is called a *complete Archimedean ordered field*. We may as well admit a certain arbitrariness in choosing this approach. **R** has been developed axiomatically in at least five other ways [e.g., as a one-dimensional continuum or as a set of binary decimals with certain arithmetic operations]. The decision to present **R** using Postulates 1 through 3 is based partly on economy and partly on personal taste.)

Postulates 1 and 2 can be used to derive all identities and inequalities which are true for real numbers [e.g., see implications (5) through (9) below]. Since arguments based on inequalities are of fundamental importance to analysis, we begin to supply details of proofs at this stage.

What is a proof? Every mathematical result (for us this includes examples, remarks, lemmas, and theorems) has hypotheses and a conclusion. There are three main methods of proof: mathematical induction, direct deduction, and contradiction.

Mathematical induction, a special method for proving statements that depend on positive integers, will be covered in Section 1.4.

To construct a *deductive proof*, we assume the hypotheses to be true and proceed step by step to the conclusion. Each step is justified by a hypothesis, a definition, a postulate, or a mathematical result that has already been proved. (Actually, this is usually the way we write a proof. When constructing your own proofs, you may find it helpful to work forward from the hypotheses as far as you can and then work backward from the conclusion, trying to meet in the middle.)

To construct a *proof by contradiction*, we assume the hypotheses to be true, the conclusion to be false, and work step by step deductively until a *contradiction* occurs; that is, a statement that is obviously false or that is contrary to the assumptions made. At this point the proof by contradiction is complete. The phrase "suppose to the contrary" always indicates a proof by contradiction (e.g., see the proof of Theorem 1.9).

What about false statements? How do we "prove" that a statement is false? We can show that a statement is false by producing a single, concrete example (called a *counterexample*) that satisfies the hypotheses but not the conclusion of that statement. For example, to show that the statement "$x > 1$ implies $x^2 - x - 2 \neq 0$" is false, we need only observe that $x = 2$ is greater than 1 but $2^2 - 2 - 2 = 0$.

Here are some examples of deductive proofs. (*Note*: The symbol ■ indicates that the proof or solution is complete.)

1.2 *EXAMPLE.*

If $a \in \mathbf{R}$, prove that

$$a \neq 0 \quad \text{implies} \quad a^2 > 0. \tag{5}$$

In particular, $-1 < 0 < 1$.

Proof. Suppose that $a \neq 0$. By the Trichotomy Property, either $a > 0$ or $a < 0$.

Case 1. $a > 0$. Multiply both sides of this inequality by a, using the First Multiplicative Property. We obtain $a^2 = a \cdot a > 0 \cdot a$. Since (by (2)), $0 \cdot a = 0$ we conclude that $a^2 > 0$.

Case 2. $a < 0$. Multiply both sides of this inequality by a. Since $a < 0$, it follows from the Second Multiplicative Property that $a^2 = a \cdot a > 0 \cdot a = 0$. This proves that $a^2 > 0$ when $a \neq 0$.

Since $1 \neq 0$, it follows that $1 = 1^2 > 0$. Adding -1 to both sides of this inequality, we conclude that $0 = 1 - 1 > 0 - 1 = -1$. ■

1.3 *EXAMPLE.*

If $a \in \mathbf{R}$, prove that

$$0 < a < 1 \quad \text{implies} \quad 0 < a^2 < a \quad \text{and} \quad a > 1 \quad \text{implies} \quad a^2 > a. \tag{6}$$

Proof. Suppose that $0 < a < 1$. Multiply both sides of this inequality by a using the First Multiplicative Property. We obtain $0 = 0 \cdot a < a^2 < 1 \cdot a = a$. In particular, $0 < a^2 < a$.

On the other hand, if $a > 1$, then $a > 0$ by Example 1.2 and the Transitive Property. Multiplying $a > 1$ by a, we conclude that $a^2 = a \cdot a > 1 \cdot a = a$. ■

Similarly (see Exercise 1.2.2), we can prove that

$$0 \le a < b \quad \text{and} \quad 0 \le c < d \quad \text{imply} \quad ac < bd, \tag{7}$$

$$0 \le a < b \quad \text{implies} \quad 0 \le a^2 < b^2 \quad \text{and} \quad 0 \le \sqrt{a} < \sqrt{b}, \tag{8}$$

and

$$0 < a < b \quad \text{implies} \quad \frac{1}{a} > \frac{1}{b} > 0. \tag{9}$$

Much of analysis deals with estimation (of error, of growth, of volume, etc.) in which these inequalities and the following concept play a central role.

1.4 Definition.

The *absolute value* of a number $a \in \mathbf{R}$ is the number

$$|a| := \begin{cases} a & a \ge 0 \\ -a & a < 0. \end{cases}$$

When proving results about the absolute value, we can always break the proof up into several cases, depending on when the parameters are positive, negative, or zero. Here is a typical example.

1.5 Remark. *The absolute value is multiplicative; that is,* $|ab| = |a|\,|b|$ *for all* $a, b \in \mathbf{R}$.

Proof. We consider four cases.

Case 1. $a = 0$ or $b = 0$. Then $ab = 0$, so by definition, $|ab| = 0 = |a|\,|b|$.

Case 2. $a > 0$ and $b > 0$. By the First Multiplicative Property, $ab > 0 \cdot b = 0$. Hence by definition, $|ab| = ab = |a|\,|b|$.

Case 3. $a > 0$ and $b < 0$, or, $b > 0$ and $a < 0$. By symmetry, we may suppose that $a > 0$ and $b < 0$. (That is, if we can prove it for $a > 0$ and $b < 0$, then by reversing the roles of a and b, we can prove it for $a < 0$ and $b > 0$.) By the Second Multiplicative Property, $ab < 0$. Hence by Definition 1.4, (2), associativity, and commutativity,

$$|ab| = -(ab) = (-1)(ab) = a((-1)b) = a(-b) = |a|\,|b|.$$

Case 4. $a < 0$ and $b < 0$. By the Second Multiplicative Property, $ab > 0$. Hence by Definition 1.4,

$$|ab| = ab = (-1)^2(ab) = (-a)(-b) = |a|\,|b|. \qquad \blacksquare$$

We shall soon see that there are more efficient ways to prove results about absolute values than breaking the argument into cases.

The following result is useful when solving inequalities involving absolute value signs.

1.6 Theorem. [Fundamental Theorem of Absolute Values]. *Let $a \in \mathbf{R}$ and $M \geq 0$. Then $|a| \leq M$ if and only if $-M \leq a \leq M$.*

Proof. Suppose first that $|a| \leq M$. Multiplying by –1, we also have $-|a| \geq -M$.

Case 1. $a \geq 0$. By Definition 1.4, $|a| = a$. Thus by hypothesis,

$$-M \leq 0 \leq a = |a| \leq M.$$

Case 2. $a < 0$. By Definition 1.4, $|a| = -a$. Thus by hypothesis,

$$-M \leq -|a| = a < 0 \leq M.$$

This proves that $-M \leq a \leq M$ in either case.

Conversely, if $-M \leq a \leq M$, then $a \leq M$ and $-M \leq a$. Multiplying the second inequality by -1, we have $-a \leq M$. Consequently, $|a| = a \leq M$ if $a \geq 0$, and $|a| = -a \leq M$ if $a < 0$. $\blacksquare$

NOTE: In a similar way we can prove that $|a| < M$ if and only if $-M < a < M$.

Here is another useful result about absolute values.

1.7 Theorem. *The absolute value satisfies the following three properties.*

i) [Positive Definite] *For all $a \in \mathbf{R}$, $|a| \geq 0$ with $|a| = 0$ if and only if $a = 0$.*
ii) [Symmetric] *For all $a, b \in \mathbf{R}$, $|a - b| = |b - a|$.*
iii) [Triangle Inequalities] *For all $a, b \in \mathbf{R}$,*

$$|a + b| \leq |a| + |b| \quad \textit{and} \quad \big|\,|a| - |b|\,\big| \leq |a - b|.$$

Proof. i) If $a \geq 0$, then $|a| = a \geq 0$. If $a < 0$, then by Definition 1.4 and the Second Multiplicative Property, $|a| = -a = (-1)a > 0$. Thus $|a| \geq 0$ for all $a \in \mathbf{R}$.

If $|a| = 0$, then by definition $a = |a| = 0$ when $a \geq 0$ and $a = -|a| = 0$ when $a < 0$. Thus $|a| = 0$ implies that $a = 0$. Conversely, $|0| = 0$ by definition.

ii) By Remark 1.5, $|a - b| = |-1|\,|b - a| = |b - a|$.

iii) To prove the first inequality, notice that $|x| \leq |x|$ holds for any $x \in \mathbf{R}$. Thus Theorem 1.6 implies that $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Adding these inequalities (see Exercise 1.2.1), we obtain

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

Hence by Theorem 1.6 again, $|a + b| \leq |a| + |b|$.

To prove the second inequality, apply the first inequality to $(a - b) + b$. We obtain

$$|a| - |b| = |a - b + b| - |b| \leq |a - b| + |b| - |b| = |a - b|.$$

By reversing the roles of a and b and applying part ii), we also obtain

$$|b| - |a| \leq |b - a| = |a - b|.$$

Multiplying this last inequality by -1 and combining it with the preceding one verifies

$$-|a - b| \leq |a| - |b| \leq |a - b|.$$

We conclude by Theorem 1.6 that $\big| |a| - |b| \big| \leq |a - b|$. ■

Notice once and for all that this last inequality implies that $|a| - |b| \leq |a - b|$ for all $a, b \in \mathbf{R}$. We will use this inequality several times.

WARNING. *Some students mistakenly mix absolute values and the Additive Property to conclude that $b < c$ implies $|a + b| < |a + c|$. It is important from the beginning to recognize that this implication is false unless both $a + b$ and $a + c$ are nonnegative.* For example, if $a = 1$, $b = -5$, and $c = -1$, then $b < c$ but $|a + b| = 4$ is not less than $|a + c| = 0$.

A correct way to estimate using absolute value signs usually involves one of the triangle inequalities.

1.8 *EXAMPLE.*

Prove that if $-2 < x < 1$, then $|x^2 - x| < 6$.

Proof. By hypothesis, $|x| < 2$. Hence by the triangle inequality and Remark 1.5,

$$|x^2 - x| \leq |x|^2 + |x| < 4 + 2 = 6.$$ ■

The following result (which is equivalent to the Trichotomy Property) will be used many times in this and subsequent chapters.

1.9 Theorem. *Let $x, y, a \in \mathbf{R}$.*

i) *$x < y + \varepsilon$ for all $\varepsilon > 0$ if and only if $x \leq y$.*
ii) *$x > y - \varepsilon$ for all $\varepsilon > 0$ if and only if $x \geq y$.*
iii) *$|a| < \varepsilon$ for all $\varepsilon > 0$ if and only if $a = 0$.*

Proof. i) Suppose to the contrary that $x < y + \varepsilon$ for all $\varepsilon > 0$ but $x > y$. Set $\varepsilon_0 = x - y > 0$ and observe that $y + \varepsilon_0 = x$. Hence by the Trichotomy Property, $y + \varepsilon_0$ cannot be greater than x. This contradicts the hypothesis for $\varepsilon = \varepsilon_0$. Thus $x \le y$.

Conversely, suppose that $x \le y$ and $\varepsilon > 0$ is given. Either $x < y$ or $x = y$. If $x < y$, then $x + 0 < y + 0 < y + \varepsilon$ by the Additive and Transitive Properties. If $x = y$, then $x < y + \varepsilon$ by the Additive Property. Thus $x < y + \varepsilon$ for all $\varepsilon > 0$ in either case. This completes the proof of part i).

ii) Suppose that $x > y - \varepsilon$ for all $\varepsilon > 0$. By the Second Multiplicative Property, this is equivalent to $-x < -y + \varepsilon$, hence by part i), equivalent to $-x \le -y$. By the Second Multiplicative Property, this is equivalent to $x \ge y$.

iii) Suppose that $|a| < \varepsilon = 0 + \varepsilon$ for all $\varepsilon > 0$. By part i), this is equivalent to $|a| \le 0$. Since it is always the case that $|a| \ge 0$, we conclude by the Trichotomy Property that $|a| = 0$. Therefore, $a = 0$ by Theorem 1.7i. ∎

Let a and b be real numbers. A *closed interval* is a set of the form

$$[a, b] := \{x \in \mathbf{R} : a \le x \le b\}, \qquad [a, \infty) := \{x \in \mathbf{R} : a \le x\},$$
$$(-\infty, b] := \{x \in \mathbf{R} : x \le b\}, \quad \text{or} \quad (-\infty, \infty) := \mathbf{R},$$

and an *open interval* is a set of the form

$$(a, b) := \{x \in \mathbf{R} : a < x < b\}, \quad (a, \infty) := \{x \in \mathbf{R} : a < x\},$$
$$(-\infty, b) := \{x \in \mathbf{R} : x < b\}, \quad \text{or} \quad (-\infty, \infty) := \mathbf{R}.$$

By an *interval* we mean a closed interval, an open interval, or a set of the form

$$[a, b) := \{x \in \mathbf{R} : a \le x < b\} \quad \text{or} \quad (a, b] := \{x \in \mathbf{R} : a < x \le b\}.$$

Notice, then, that when $a < b$, the intervals $[a, b]$, $[a, b)$, $(a, b]$, and (a, b) correspond to line segments on the real line, but when $b < a$, these "intervals" are all the empty set.

An interval I is said to be *bounded* if and only if it has the form $[a, b]$, (a, b), $[a, b)$, or $(a, b]$ for some $-\infty < a \le b < \infty$, in which case the numbers a, b will be called the *endpoints* of I. All other intervals will be called *unbounded*. An interval with endpoints a, b is called *degenerate* if $a = b$ and *nondegenerate* if $a < b$. Thus a degenerate open interval is the empty set, and a degenerate closed interval is a point.

Analysis has a strong geometric flavor. Geometry enters the picture because the real number system can be identified with the real line in such a way that $a < b$ if and only if a lies to the left of b (see Figures 1.2, 2.1, and 2.2). This gives us a way of translating analytic results on $\mathbf{R}$ into geometric results on the number line, and vice versa. We close with several examples.

The absolute value is closely linked to the idea of length. The *length* of a bounded interval I with endpoints a, b is defined to be $|I| := |b - a|$, and the *distance* between any two points $a, b \in \mathbf{R}$ is defined by $|a - b|$.

Inequalities can be interpreted as statements about intervals. By Theorem 1.6, $|a| \le M$ if and only if a belongs to the closed interval $[-M, M]$; and by Theorem 1.9, a belongs to the open interval $(-\varepsilon, \varepsilon)$ for all $\varepsilon > 0$ if and only if $a = 0$.

We will use this point of view in Chapters 2 through 5 to give geometric interpretations to the calculus of functions defined on $\mathbf{R}$, and in Chapters 11 through 13 to extend this calculus to functions defined on the Euclidean spaces $\mathbf{R}^n$.

EXERCISES

In each of the following exercises, verify the given statement carefully, proceeding step by step. Validate each step that involves an inequality by using some statement found in this section.

1.2.0 Let $a, b, c, d \in \mathbf{R}$ and consider each of the following statements. Decide which are true and which are false. Prove the true ones and give counterexamples to the false ones.

a) If $a < b$ and $c < d < 0$, then $ac > bd$.
b) If $a \le b$ and $c > 1$, then $|a + c| \le |b + c|$.
c) If $a \le b$ and $b \le a + c$, then $|a - b| \le c$.
d) If $a < b - \varepsilon$ for all $\varepsilon > 0$, then $a < 0$.

1.2.1. Suppose that $a, b, c \in \mathbf{R}$ and $a \le b$.

a) Prove that $a + c \le b + c$.
b) If $c \ge 0$, prove that $a \cdot c \le b \cdot c$.

1.2.2. Prove (7), (8), and (9). Show that each of these statements is false if the hypothesis $a \ge 0$ or $a > 0$ is removed.

1.2.3. **This exercise is used in Section 6.3**. The *positive part* of an $a \in \mathbf{R}$ is defined by

$$a^+ := \frac{|a| + a}{2}$$

and the *negative part* by

$$a^- := \frac{|a| - a}{2}.$$

a) Prove that $a = a^+ - a^-$ and $|a| = a^+ + a^-$.
b) Prove that

$$a^+ = \begin{cases} a & a \ge 0 \\ 0 & a \le 0 \end{cases} \quad \text{and} \quad a^- = \begin{cases} 0 & a \ge 0 \\ -a & a \le 0. \end{cases}$$

1.2.4. Solve each of the following inequalities for $x \in \mathbf{R}$.

a) $|4x - 2| < 22$
b) $|1 - 2x| < 7$

c) $|x^3 - x| < x^3$
d) $\dfrac{2x}{x-2} < 4$
e) $\dfrac{3x^2}{3x^2-3} < 1$

1.2.5. Let $a, b \in \mathbf{R}$.

a) Prove that if $a > 2$ and $b = 1 + \sqrt{a-1}$, then $2 < b < a$.
b) Prove that if $2 < a < 3$ and $b = 2 + \sqrt{a-2}$, then $0 < a < b$.
c) Prove that if $0 < a < 1$ and $b = 1 - \sqrt{1-a}$, then $0 < b < a$.
d) Prove that if $3 < a < 5$ and $b = 2 + \sqrt{a-2}$, then $3 < b < a$.

1.2.6. The *arithmetic mean* of $a, b \in \mathbf{R}$ is $A(a,b) = (a+b)/2$, and the *geometric mean* of $a, b \in [0, \infty)$ is $G(a,b) = \sqrt{ab}$. If $0 \le a \le b$, prove that $a \le G(a,b) \le A(a,b) \le b$. Prove that $G(a,b) = A(a,b)$ if and only if $a = b$.

1.2.7. Let $x \in \mathbf{R}$.

a) Prove that $|x| \le 4$ implies $|x^2 - 1| \le 5\,|x-1|$.
b) Prove that $|x| \le 1$ implies $|x^2 + 5x - 6| \le 7\,|x-1|$.
c) Prove that $-3 \le x \le 3$ implies $|x^2 + 3x - 10| \le 8\,|x-2|$.
d) Prove that $-2 < x < 0$ implies $|x^3 + 3x^2 - 6x - 8| < 9.5\,|x+1|$.

1.2.8. For each of the following, find all values of $n \in \mathbf{N}$ that satisfy the given inequality.

a) $\dfrac{1-2n}{1-4n^2} < 0.02$
b) $\dfrac{n^3-1}{n^3+n^2+n} < 2.5$
c) $\dfrac{n-2}{n^3-2n^2+4n-8} < 0.001$

1.2.9. a) Interpreting a rational m/n as $m \cdot n^{-1} \in \mathbf{R}$, use Postulate 1 to prove that

$$\frac{m}{n} + \frac{p}{q} = \frac{mq+np}{nq}, \quad \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}, \quad -\frac{m}{n} = \frac{-m}{n}, \text{ and } \left(\frac{\ell}{n}\right)^{-1} = \frac{n}{\ell}$$

for $m, n, p, q, \ell \in \mathbf{Z}$ and $n, q, \ell \ne 0$.
b) Using Remark 1.1, Prove that Postulate 1 holds with $\mathbf{Q}$ in place of $\mathbf{R}$.
c) Prove that the sum of a rational and an irrational is always irrational. What can you say about the product of a rational and an irrational?
d) Let $m/n, p/q \in \mathbf{R}$ with $n, q > 0$. Prove that

$$\frac{m}{n} < \frac{p}{q} \quad \text{if and only if} \quad mq < np.$$

(Restricting this observation to $\mathbf{Q}$ gives a definition of "$<$" on $\mathbf{Q}$.)

1.2.10. Prove that

$$(ab + cd)^2 \le (a^2 + c^2)(b^2 + d^2)$$

for all $a, b, c, d \in \mathbf{R}$.

1.2.11. a) Let $\mathbf{R}^+$ represent the collection of positive real numbers. Prove that $\mathbf{R}^+$ satisfies the following two properties.

i) For each $x \in \mathbf{R}$, one and only one of the following holds:

$$x \in \mathbf{R}^+, \quad -x \in \mathbf{R}^+, \quad \text{or} \quad x = 0.$$

ii) Given $x, y \in \mathbf{R}^+$, both $x + y$ and $x \cdot y$ belong to $\mathbf{R}^+$.

b) Suppose that $\mathbf{R}$ contains a subset $\mathbf{R}^+$ (not necessarily the set of positive numbers) which satisfies properties i) and ii). Define $x \prec y$ by $y - x \in \mathbf{R}^+$. Prove that Postulate 2 holds with $\prec$ in place of $<$.

1.3 COMPLETENESS AXIOM

In this section we introduce the last of three postulates that describe $\mathbf{R}$. To formulate this postulate, which distinguishes $\mathbf{Q}$ from $\mathbf{R}$, we need the following concepts.

1.10 Definition.

Let $E \subset \mathbf{R}$ be nonempty.

i) The set E is said to be *bounded above* if and only if there is an $M \in \mathbf{R}$ such that $a \le M$ for all $a \in E$, in which case M is called an *upper bound* of E.

ii) A number s is called a *supremum* of the set E if and only if s is an upper bound of E and $s \le M$ for all upper bounds M of E. (In this case we shall say that *E has a finite supremum s* and write $s = \sup E$.)

NOTE: Because French mathematicians (e.g., Borel, Jordan, and Lebesgue) did fundamental work on the connection between analysis and set theory, and *ensemble* is French for *set*, analysts frequently use E to represent a general set.

By Definition 1.10ii, a supremum of a set E (when it exists) is the smallest (or least) upper bound of E. By definition, then, in order to prove that $s = \sup E$ for some set $E \subset \mathbf{R}$, we must show two things: s is an upper bound, AND s is the smallest upper bound. Here is a typical example.

1.11 *EXAMPLE.*

If $E = [0, 1]$, prove that $\sup E = 1$.

Proof. By the definition of interval, 1 is an upper bound of E. Let M be any upper bound of E; that is, $M \ge x$ for all $x \in E$. Since $1 \in E$, it follows that $M \ge 1$. Thus 1 is the smallest upper bound of E. ■